YOUR FOOD IS ALWAYS OUTSIDE OF YOU

(Some Ideas About Space But Definitely Not About Time)

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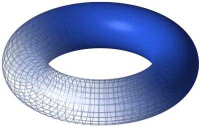

Figure 1 : Me IRL. [1]

Introduction

- Let's talk about some things!
- Ask questions
- OK if you get lost, we'll move on in a few slides
- Stop me when it's time for dinner

- References at the end of these slides: `http://iank.org/ncsulug_fa14.pdf`
- Blog: `http://blog.iank.org/`
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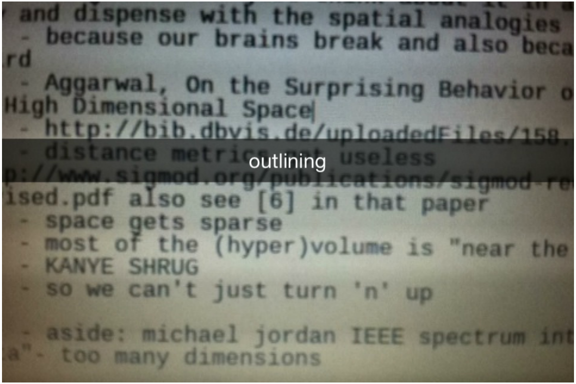
Outline

Inverse Square Laws

High Dimensional Space

Coordinate and Mathematical Spaces



Let's talk about Kant!

- J.D. Barrow, *The Constants of Nature* [2]
- Immanuel Kant, 1747, *Thoughts on the True Estimation of Living Forces . . .* [3]
- Kant's first published work
- Maybe the first to wonder about connection between Newton's gravitation and 3-D space
- Got it backwards though, so he's a philosopher now

Newton's Law of Universal Gravitation

Attractive force due to gravity:

$$F = G\frac{m_1 m_2}{r^2}$$

Chill out:

$$F \propto \frac{1}{r^2}$$

- Effect falls off by r^2
- True for gravity, electromagnetism, acoustics, ...
- Why 2?
- Claim: $F \propto r^{-2}$ *because* space is 3-D
- Gauss, stokes, ..., or:

A Nice Illustration

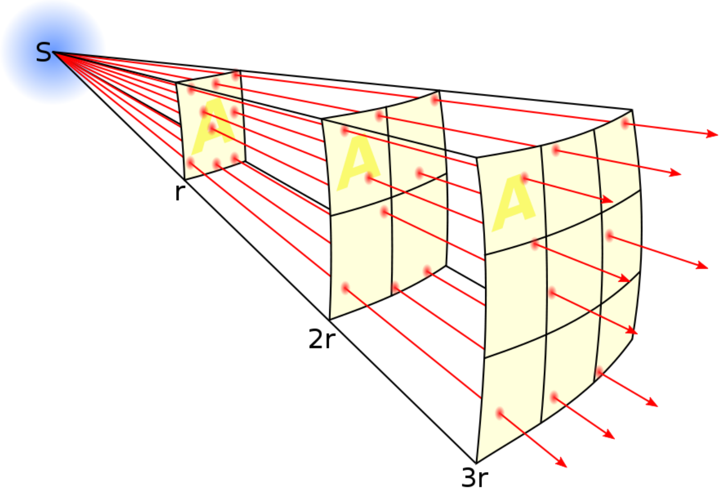


Figure 2 : Inverse square law for an isotropic point source [4]

Kant's argument

- [3] (Section 9): space would not exist "if substances had no forces to act external to themselves"
- [3] (Section 10): 3-D space is a consequence of inverse-square gravity
- (Exactly backwards, but we'll give it to him!)

inverse square law is a *mathematical consequence* of 3-D space.

Stable orbits

- among the thousand other things that are apparently coincidentally Just Exactly Right for us in this universe
- Stable orbits only exist in 2D or 3D space

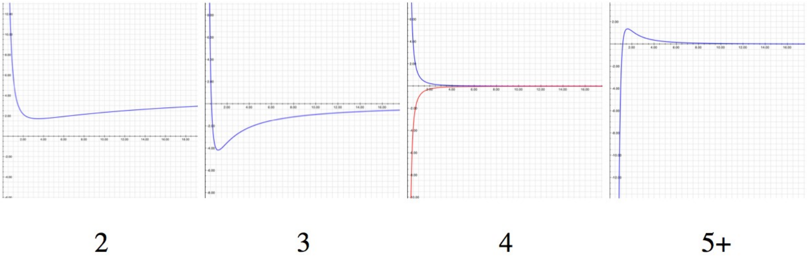


Figure 3 : Gravitational vs centrifugal energy potential in n-D [5]

Barrow's theme

- We haven't described all of physics
- Our universe is in a precariously narrow range of values of a few different constants
- Chemistry, life, etc, breaks if we move in any direction
- Luck? God? Anthropic principle? (multiverse, multiple domains/inflation)
- Some constants are derived (e.g. elementary charge)
- We will likely find that more of them are derived
- Maybe there is only one system of Physics w/ all of these values fixed by internal consistency

ASIDE: Nonexhaustive List of Other Things That Depend on the Dimension

- Chemistry
- Knots
- Everything
- Wave propagation (esp. $2n$) [6]
- Rotation???
- Polya: random walk on integer lattice [7]
- 2-D digestive tract [8]
- `<@krrrlson> YOUR FOOD IS ALWAYS OUTSIDE OF YOU`

Diaspora

For a fictional but cool take on this and some other things, see *Diaspora* by Greg Egan [9].

JOKE BREAK

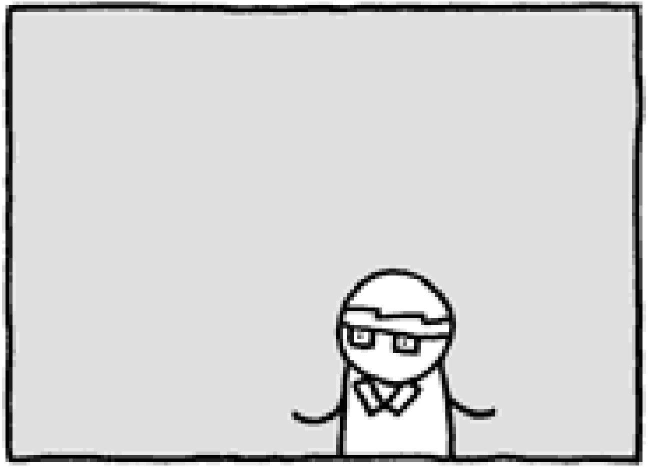


Figure 4 : IT CAN BE SHOWN [10]

How I Learned To Stop Worrying And Love Linear Algebra

(DRAW A 4-D PICTURE AT THIS PART)

- Graphics and spatial reasoning only take us so far
- 4-D is okay but ≥ 5 breaks my mind
- Mathematical abstraction:

Vector in $3\mathrm{D}$:

$$(x, y, z)$$

Vector in $n\mathrm{D}$:

$$(x_1, x_2, \ldots, x_n)$$

Things Get Super Weird (Curse of Dimensionality)

- Data becomes sparse
- Distance metrics stop working
- Volume is "near the edges"

Sparse Data

- combinatorial explosion (consider binary)
- volume increases rapidly w/r/t to data
- 10 evenly-spaced points on unit interval $\Rightarrow 10^2$ points on unit square $\Rightarrow 10^{10}$ points in 10-dimensional space

Distance Metrics

- Remember 2-norm?

$$\|\mathbf{p_1} - \mathbf{p_2}\| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

- "Relative contrast" $\rightarrow 0$ as $d \rightarrow \infty$ [11]
- Breaks clustering, search (indexes) [12], also everything else
- True for other distance metrics as well [13]
- finding better distance metrics only works so far [14]
- distance metrics may not be **qualitatively** meaningful in higher dimensions [11]
- sometimes we can re-design problem:

One approach: Dimensionality Reduction

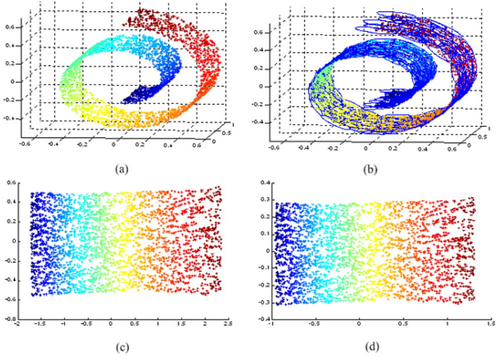


Figure 5 : Swiss Roll Manifold. After [15]

- Some high-dimensional data has low-dimensional structure
- Use dimensionality reduction techniques
  - hand-crafted features
  - PCA, e.g. “Eigenfaces” [16]
  - Others, e.g. t-SNE [17]

Volume "near the edges"

Volume of a d-hypersphere [18]

$$V_{sphere}(d) = \frac{r^d \pi^{d/2}}{\Gamma(1 + \frac{d}{2})}$$

Volume of a d-hypercube

$$V_{cube}(d) = (2r)^d$$

Limit of the ratio

$$\lim_{d \to \infty} \frac{V_{sphere}(d)}{V_{cube}(d)} = 0$$

Further: "All of the [hypersphere's] volume is in the crust" [19]

High-dimensional gaussian

[19]: as $d \to \infty$, probability mass in the tails

BIG DATA

- Dan Ariely's joke
- Michael Jordan has Good Opinions [20]
- Football example
- See Goldacre, Silver, Mlodinow [21, 22, 23]

JOKE BREAK

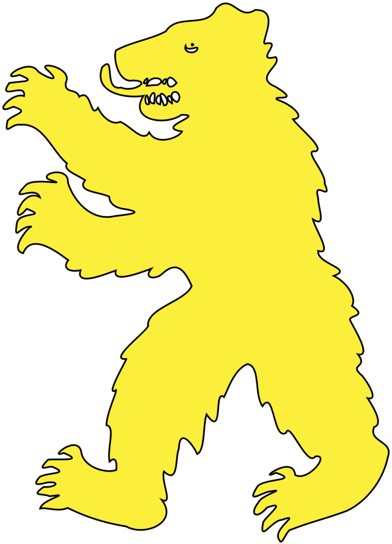


Figure 6 : Not This Colour [24]

Let's talk about Cthulhu

"He had said that the geometry of the dream-place he saw was abnormal, non-Euclidean, and loathsomely redolent of spheres and dimensions apart from ours."
– H. P. Lovecraft, The Call of Cthulhu [25]

non-definition of Euclidian space:

- Parallel lines extend forever without meeting
- Congruence/similarity – translation/rotation/reflection/scaling
- Objects can be moved without deformation
- $\mathbb{R}^n$

ASIDE

General Relativity: actual space is non-Euclidian, h/t Lovecraft

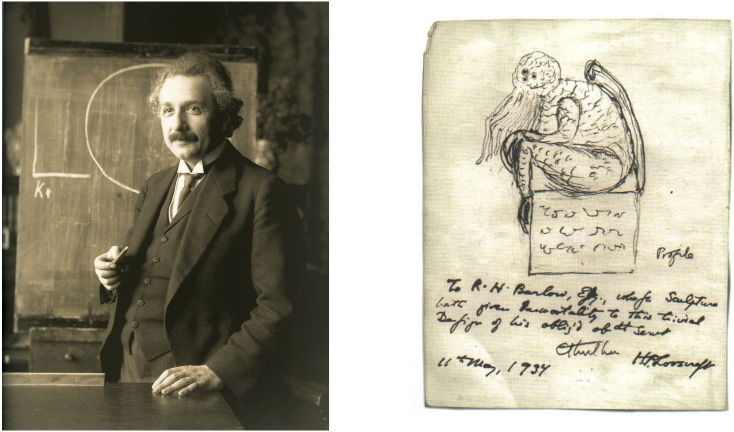


Figure 7 : Albert Einstein, assistant examiner at the R'lyeh Patent Office

Spherical Geometry

- [26] emphasizes difference between coordinate systems and mathematical spaces
- Consider the 2-D surface of a 3-D sphere
  - *elliptic 2-space*
  - lines $\rightarrow$ great circles
  - closed, finite
  - translation without deformation
  - not scale-invariant
  - no parallel lines (all great circles intersect)

Spherical Triangles

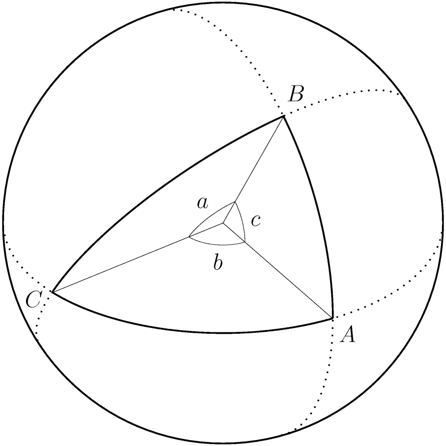


Figure 8 : Great Circle Triangle on a Sphere. After [27]

- Sum of angles: $180^\circ \cdots 540^\circ$
- Spherical trigonometry

Pythagoreas

Euclidian geometry:

$$c^2 = a^2 + b^2$$

Elliptic geometry:

$$cos\left(\frac{c}{R}\right) = cos\left(\frac{a}{R}\right) cos\left(\frac{b}{R}\right)$$

(Special case of Spherical Law of Cosines)

Claim: for extremely small shapes (or, equivalently, spheres with extremely large R), elliptic space appears Euclidian

Not a proof (1/2)

Take limit as $R \to \infty$:

$$cos\left(\frac{c}{R}\right) = cos\left(\frac{a}{R}\right) cos\left(\frac{b}{R}\right)$$

MacLaurin series:

$$cos(x) \approx 1 - \frac{x^2}{2} + \cdots \quad \mathrm{as\ x} \to 0$$

This is a valid approximation b/c:

$$\lim_{R\to\infty} \frac{c}{R} = 0,$$

so substitute:

$$\left[1 - \frac{1}{2}\left(\frac{c}{R}\right)^2 + \cdots\right] = \left[1 - \frac{1}{2}\left(\frac{a}{R}\right)^2 + \cdots\right]\left[1 - \frac{1}{2}\left(\frac{b}{R}\right)^2 + \cdots\right]$$

Not a proof (2/2)

Collect terms

$$\left(\frac{c}{R}\right)^2 = \left(\frac{a}{R}\right)^2 + \left(\frac{b}{R}\right)^2 - \frac{1}{2}\frac{a^2 b^2}{R^4} + \cdots$$

neglect everything R^{-4} or smaller:

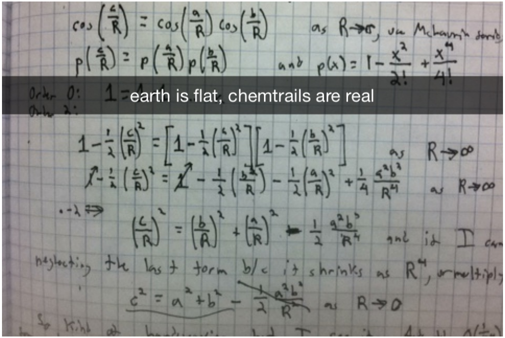
$$\left(\frac{c}{R}\right)^2 = \left(\frac{a}{R}\right)^2 + \left(\frac{b}{R}\right)^2$$

cancel R^{-2}:

$$c^2 = a^2 + b^2$$

Differential Geometry

- Sphere appears Euclidian in the differential limit
- (IOW, elliptic space is differentially flat)
- ∴ (small) triangles still work on Earth and other kinds of bears don't get lost
- not true for other spaces
- "egg" is always curved in one direction, even differentially



Conclusion

- Barrow $\rightarrow$ Kant $\rightarrow$ Physics $\rightarrow$ Dimensionality
- Lots of those $\rightarrow$ Big Problems
- Non-Euclidian spaces $\rightarrow$ Earth is flat

Figure 9 : You can do strained metaphors in any dimension [28]

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