Nonlinear Dynamics and Chaos

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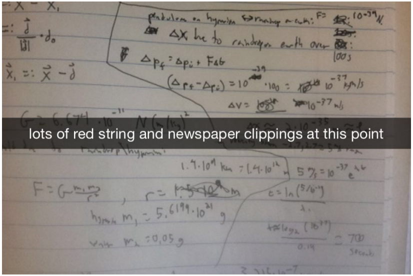
2015-04-07



Resources

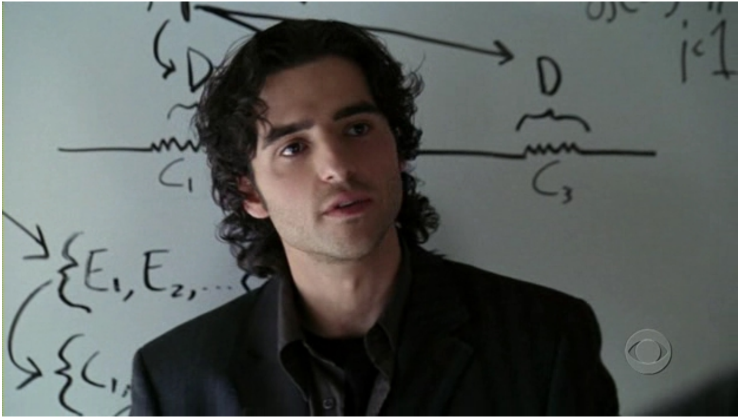
Three books and a number of papers (references at the end)

- Kellert, *In the Wake of Chaos*. 1994
- Gleick, *Chaos: Making a New Science*. 1987
- Sternberg, *Dynamical Systems*. 2010

Links:

- These slides are online: `http://iank.org/ncsulug_sp15.pdf`
- Code: `http://github.com/iank/`
- email: `iank@iank.org`

“Chaos Theory”



The Seagull Effect

“Lorenz originally used the image of a seagull.” (Gleick, 1987, p. 329).

I'm going to make a ridiculous claim and prove it later

> *"The basic idea of Western science is that you don't have to take into account the falling of a leaf on some planet in another galaxy when you're trying to account for the motion of a billiard ball on a pool table on earth."*

– Arthur Winfree, in (Gleick, 1987, p. 14).

Linear systems

Operator $H\{\cdot\}$:

$$y(t) = H\{x(t)\}$$

Obeys superposition (Pedro & Carvalho, 2002, p. 6):

$$H\{a \cdot x_1(t) + b \cdot x_2(t)\} = a \cdot H\{x_1(t)\} + b \cdot H\{x_2(t)\}$$

Linear systems are easy

Simple harmonic motion:

$$\frac{d^2x}{dx^2} + \frac{k}{m} \cdot x = 0$$

Solution:

$$x(t) = A\cos(\omega t - \varphi),$$
$$\omega = \sqrt{\frac{k}{m}}$$

Nonlinear systems are hard

Simple model for a nonlinear pendulum

$$\frac{d^2\varphi}{d\varphi^2} + \frac{g}{\ell} \cdot sin(\varphi) = 0$$

Nonlinear systems are hard

Simple model for a nonlinear pendulum

$$\frac{d^2\varphi}{d\varphi^2} + \frac{g}{\ell} \cdot sin(\varphi) = 0$$

Solution (Ochs, 2011):

$$\begin{aligned}
\theta(t) &= sgn(\dot{\varphi}_0) k\Omega \left[t - t_0\right] + \mathrm{sn}^{-1}(k_0|x), \\
\varphi(t) &= 2 \arcsin(\mathrm{sn}(\theta(t)|x))\mathrm{sgn}(\mathrm{cn}(\theta(t)|x)), \\
\dot{\varphi}(t) &= \mathrm{sgn}(\dot{\varphi}_0)\sqrt{E_0}\mathrm{dn}(\theta(t)|x).
\end{aligned}$$

State of systems can be represented as a space

- Simple harmonic motion
- Nonlinear pendulum

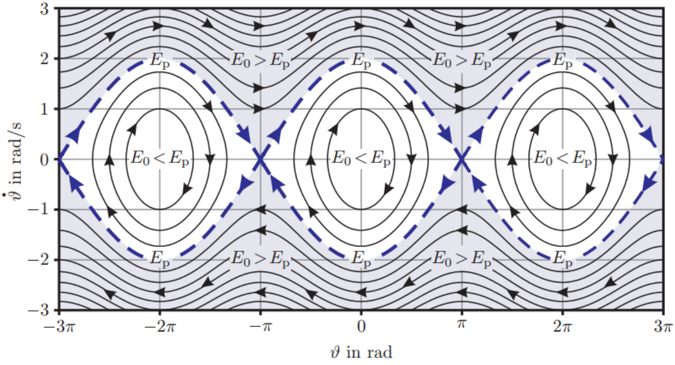


Figure: Phase space for nonlinear pendulum. After (Ochs, 2011)

Attractors - fixed point

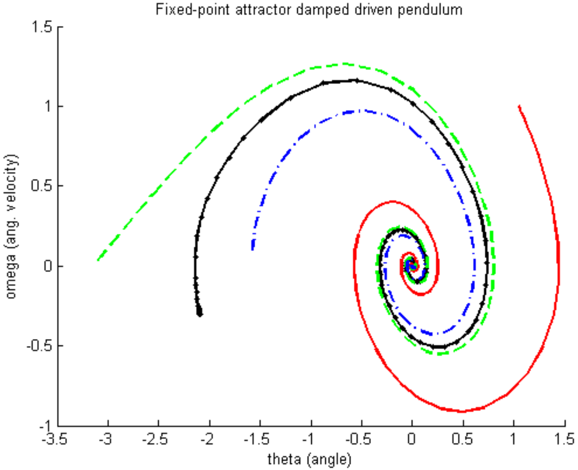


Figure: Trivial fixed-point attractor for damped pendulum. Pictured: various initial positions and velocities

[1] https://github.com/iank/pendulums see damped_fixedpoint.m

Attractors - limit cycle

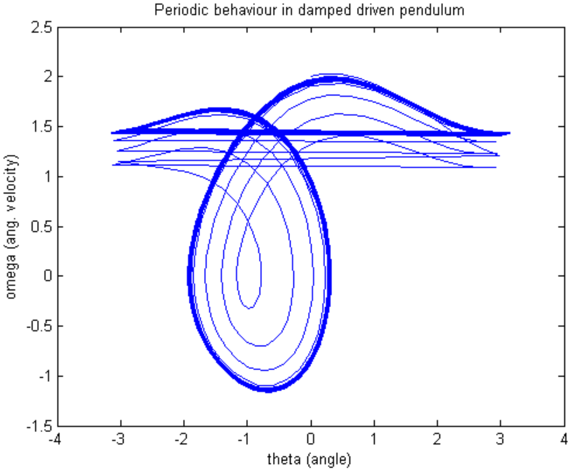


Figure: Transient + periodic behaviour in damped driven pendulum

[1] https://github.com/iank/pendulums see damped_driven_pendulum.m

Multiple attractors - basins

$f(x) = x^3 - 1$ has three complex roots. Use Newton's method (Sternberg, 2010, p. 20)

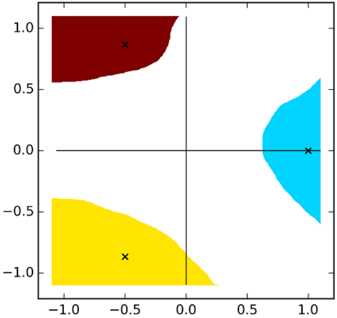


Figure: Three solutions and partial attractive basins for Newton's Method on $x^3 - 1$ in the complex plane

[1] https://github.com/iank/newton_basins

Fractal basin boundaries (1/3)

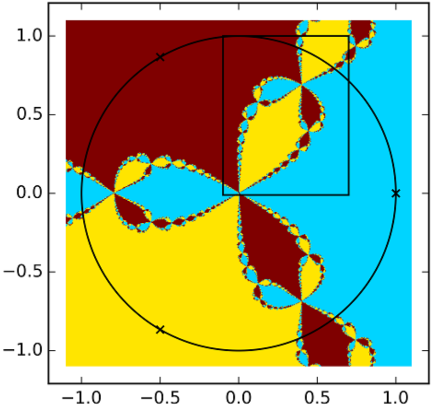


Figure: Three solutions and attractive basins for Newton's Method on $x^3 - 1$ in the complex plane

[1] https://github.com/iank/newton_basins

Fractal basin boundaries (2/3)

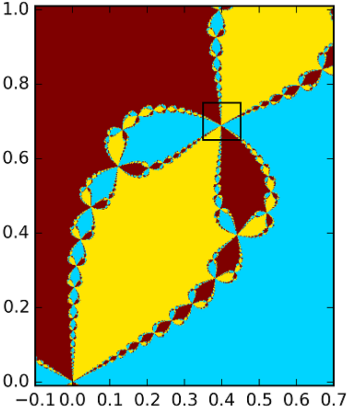


Figure: Fractal basins for Newton's Method on $x^3 - 1$ in the complex plane

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Fractal basin boundaries (3/3)

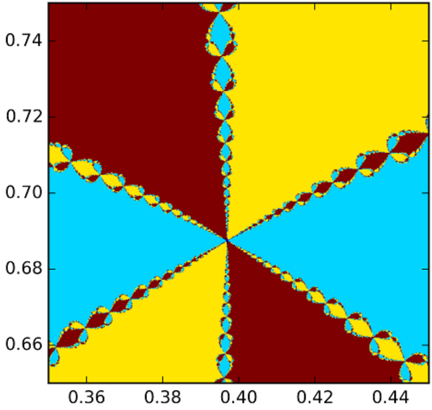
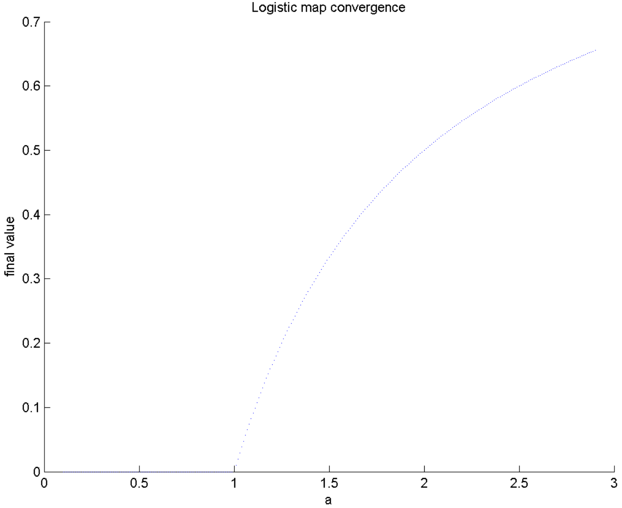


Figure: Self-similarity in fractal boundary

[1]https://github.com/iank/newton_basins

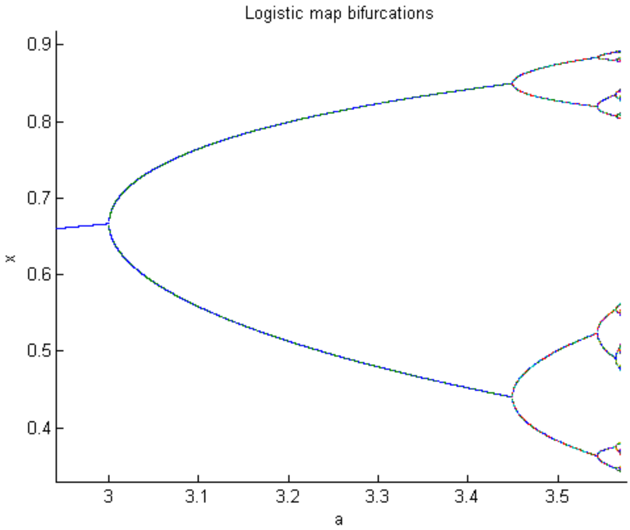
Period-doubling / bifurcations - (1/3)

Ex. Logistic Map (May et al., 1976)



[1] https://github.com/iank/logistic_map_bifurcation

Period-doubling / bifurcations - (2/3)



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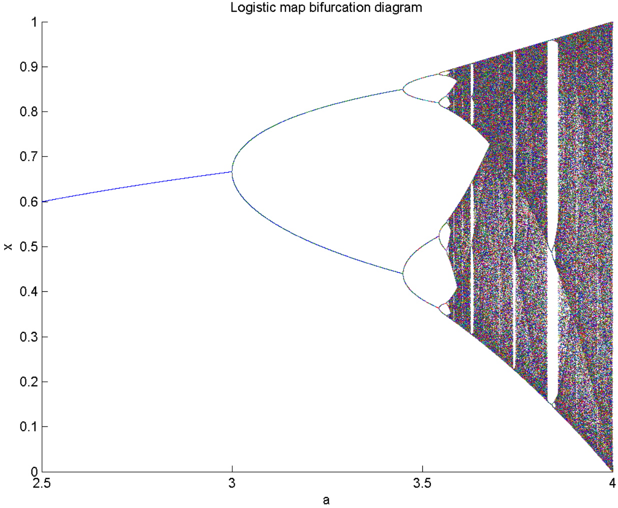

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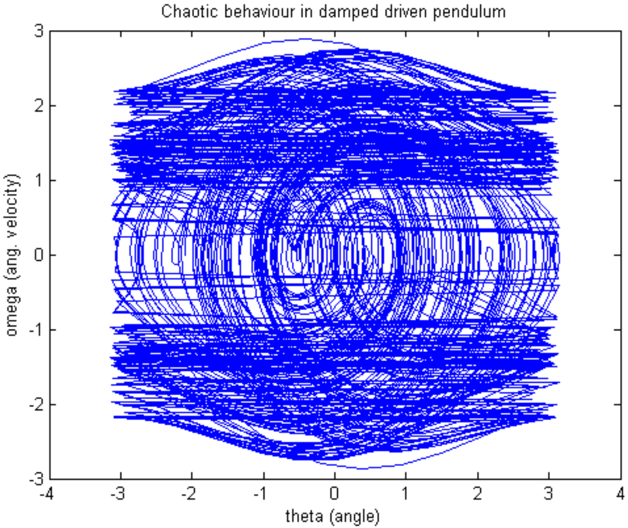
Chaos



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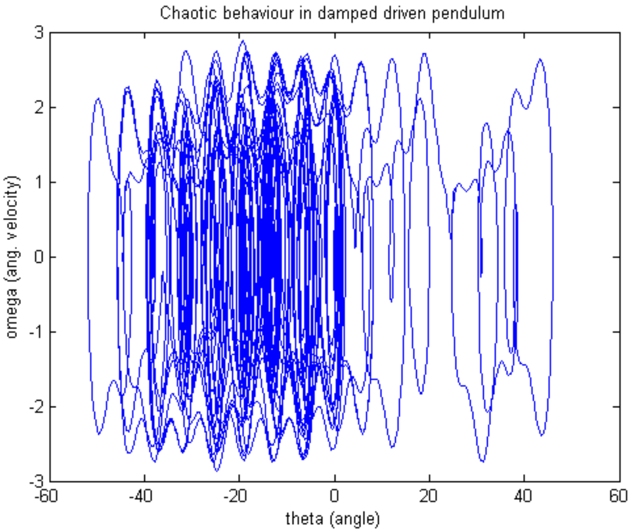
Chaos



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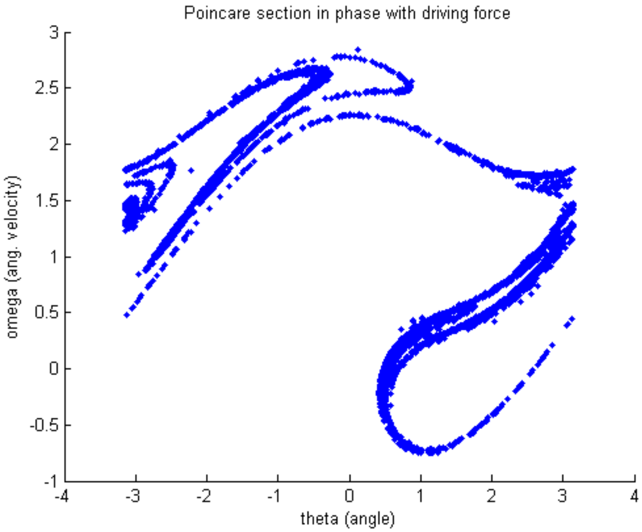
MATLAB BREAK

Strange attractors



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Stretching and folding

Various formal definitions of chaos (Sternberg, 2010, p.84)

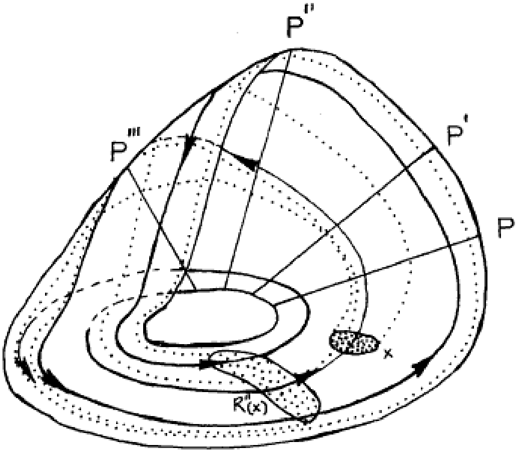
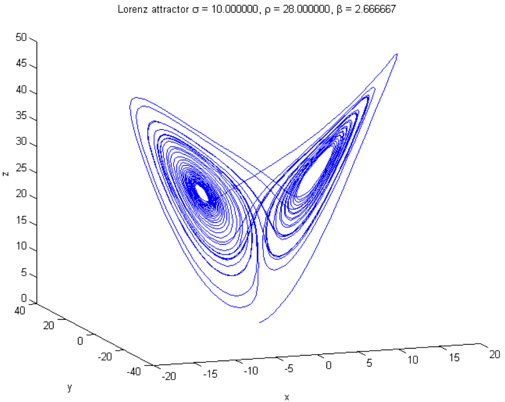


Figure: Stretching and folding in the Rössler attractor. After (Schaffer, 1984).

Topological mixing - (1/2)

Lorenz system (Lorenz, 1963)

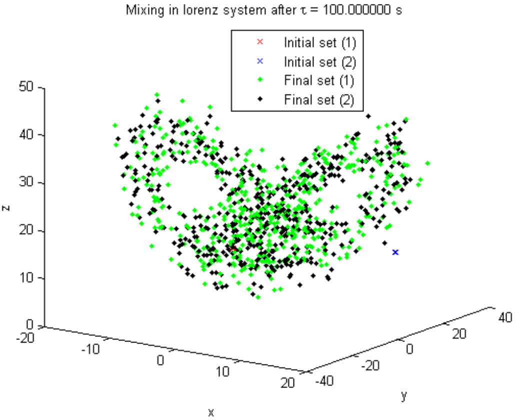
$$\frac{dx}{dt} = \sigma(y - x), \frac{dy}{dt} = x(\rho - z) - y, \frac{dz}{dt} = xy - \beta z.$$



[1]https://github.com/iank/lorenz_mixing see lorenz.m

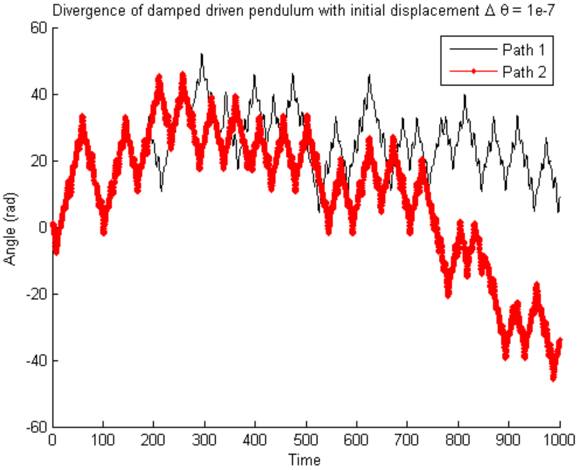
MATLAB BREAK

Topological mixing - (2/2)



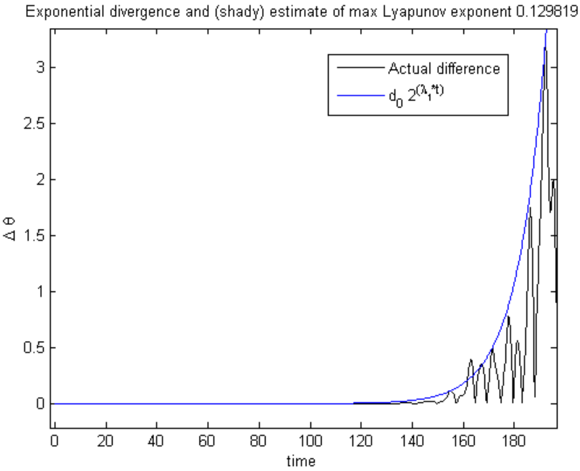
[1] https://github.com/iank/lorenz_mixing see lorenz_mixing.m

Limits on predictibility of systems - (1/2)



[1] https://github.com/iank/pendulums see pendulum_divergence.m

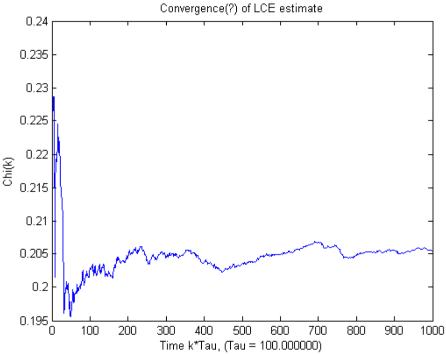
Limits on predictibility of systems - (2/2)



[1] https://github.com/iank/pendulums see pendulum_divergence_exponential.m

Lyapunov

- Difficult to estimate Lyapunov time empirically (Tancredi, Sánchez, & Roig, 2001).
- My estimate based on renormalization method in (Benettin, Galgani, Giorgilli, & Strelcyn, 1980): 0.20
- Reasonable for this system (Wolf, Swift, Swinney, & Vastano, 1985)



[1] https://github.com/iank/pendulums see lyapunov.m

Leaf falling on another planet.. - (1/3)

Consider a leaf in a tree on earth and a damped, driven pendulum on Hyperion

- Conservative estimate of LCE: $\lambda_1 = 0.12$ bit/s
- $m_{leaf} = 0.1g$
- Hyperion is about 1,200 gigameters away from the leaf
- Leaf falls

Leaf falling on another planet.. - (2/3)

- Hyperion is about 1,200,000,000,000 + 10 m away from the leaf
- Acceleration of a mass due to gravity: $a_{grav} = G\frac{m_{leaf}}{r^2}$
- Difference in gravitational acceleration due to leaf in tree vs leaf on ground $\approx 7.7 * 10^{-50} \approx 10^{-51} m/s^2$
- $\Delta v \approx a_{grav} * \Delta t$
- Let $\Delta t = 10s$ (!!)
- $\Delta v \approx 10^{-50} \frac{m}{s}$

Leaf falling on another planet.. - (3/3)

- 10 m/s is enough uncertainty to have no idea where the pendulum is
- Lyapunov defn: $10 = 10^{-50} \cdot 2^{\lambda_1 t}$
- Solve: $\log_2 \frac{10}{10^{-50}} / 0.12 = t$

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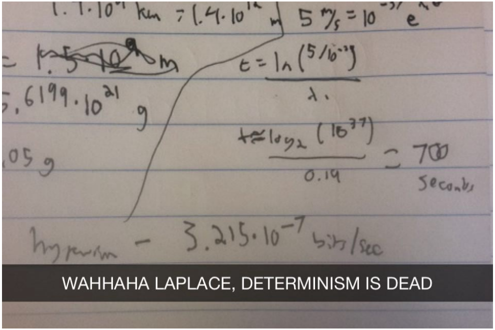
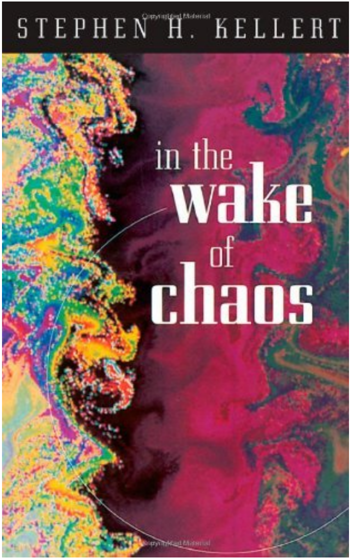


Kellert - (1/4)



(Kellert, 1994, ch. 2, 3)

Kellert - (2/4) - Varieties of the impossible

Types of impossibility: (Kellert, 1994, ch. 2)

1. logical
2. theoretical
3. practical: "completion would require more resources than are available to human beings"

   [practical] impossibility does not hold for all times and places

– (Kellert, 1994, p. 37).

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– (Kellert, 1994, p. 37).
Kellert upgrades prediction of chaotic systems from practical to theoretical

Kellert - (3/4) - Determinism

Senses of “deterministic”: (Kellert, 1994, pp. 57-61)

1. Differential dynamics
2. Unique evolution (Laplacian)
3. Value determinateness
4. Total predictability

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Check: (Kellert, 1994, pp. 62-71)

4. Unique trajectory may exist but we can never know which a system is on
3. $\Delta x \Delta p > \frac{\hbar}{2}$
2. Yes for classical and non-dissipative systems. No if we turn on QM.

Kellert - (4/4) - Identical worlds

- Unique evolution: "if there were two identical worlds at time t_0, then they would be identical at all other times" (Kellert, 1994, p.74)
- What does "identical world" mean? Either:
  - "all particles have the same position, momentum, etc, even to an infinite number of decimal places"
  - "the same, so far as physics can specify"

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Kellert: Chaos + QM: "stuff happens. It just happens."

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Kellert: Chaos + QM: "stuff happens. It just happens."
Local determinism (Kellert, 1994, p. 75)

Chaos allows us to make predictions in apparently disordered systems

Winfree's mosquito anecdote (Gleick, 1987, p. 285)
Measles (Schaffer, 1984), ecology (Schaffer & Kot, 1985)

BONUS SLIDE

- Many-body problem, stability of solar system (Laskar & Gastineau, 2009)
- Universality (Feigenbaum, 1983)
- "Period 3 implies chaos" (Li & Yorke, 1975)
- Soviets, feminism, linear bias, digital computers (Kellert, 1994, ch. 5) vs (Gleick, 1987)
- Von Neuman weather control (Gleick, 1987, p. 18)
- Snowflakes! (Gleick, 1987, pp. 309-311),
- Reconstruction of phase space from experimental data (Schaffer, 1984)
- Scale
- High-dimensional chaos
- White earth climate (Gleick, 1987, p. 170)

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